# Functional Analysis

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**Exercise 0.1.** The sequence space $\ell^p$ for $1 \le p \le \infty$ consists of all sequences $x = (x_n)_{n=1}^{\infty}$ of scalars such that:

$$\|x\|_p = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \sup_{n\in\mathbb{N}} |x_n| & \text{if } p = \infty, \end{cases}$$

is finite. Each $\ell^p$ space is a Banach space.

**Exercise 0.2.** The space $L^p([a,b])$ for $1 \le p \le \infty$ consists of (equivalence classes of) measurable functions $f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$) such that the $p$-th power of the absolute value is integrable:

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \text{ess sup}_{x\in[a,b]} |f(x)| & \text{if } p = \infty. \end{cases}$$

These spaces are Banach spaces.

**Exercise 0.3.** The space $C([a,b])$ of continuous real (or complex) functions on $[a,b]$ equipped with the *supremum norm*

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$$

is also a Banach space.

**Exercise 0.4** (Non-complete normed space)**.** Let $c_{00}$ denote the space of sequences with only finitely many nonzero terms, equipped with the $\ell^p$ norm for some $1 \le p < \infty$. Then $(c_{00}, \|\cdot\|_p)$ is a normed space, but it is not complete — its completion is $\ell^p$.

**Exercise 0.5.** Consider the space of polynomials $P([0,1])$ with the sup norm. Is this a Banach space?

**Exercise 0.6.** Let $X = c$, the space of convergent sequences with the sup norm. Show that $X$ is a Banach space.

**Exercise 0.7.** Define $T : \ell^2 \to \ell^2$ by $T(x_1, x_2, x_3, ...) = (x_1, x_2/2, x_3/3, ...)$. Prove that $T$ is bounded.

**Exercise 0.8.** Let $T$ be defined on $L^2([0,1])$ by $(Tf)(x) = \int_0^x f(t)dt$. Show that $T$ is a bounded linear operator.

**Exercise 0.9.** Let $(X, \|\cdot\|)$ be a normed space. For every $x_0 \in X$ there exists a continuous functional $f_0 : X \to \mathbb{R}$ such that $\|f_0\| = \|x_0\|$ and $f_0(x_0) = \|x_0\|^2$.