Functional Analysis

Jarosław Mederski

Scuola Matematica Interuniversitaria, Perugia 2025

Table of Contents

Exercise 0.1. Let c_0 be the space of all sequences converging to zero, equipped with the supremum norm. Is c_0 a closed subspace of ℓ^∞? Is it a Banach space?

Exercise 0.2. Let $X = \mathbb{R}^2$ and $M = \text{span}((1,1))$. Define the quotient space X/M. Describe the equivalence classes and the geometry of this space.

Exercise 0.3. Let $X = \ell^2$ and let $M \subset X$ be the subspace consisting of all sequences with only the first coordinate possibly nonzero. Describe the quotient space X/M. Is it a Banach space?

Exercise 0.4. Find the dual space $(\ell^1)^*$. Prove that

$$(\ell^1)^* \cong \ell^\infty.$$

Exercise 0.5. Define the operator $T : \ell^2 \to \ell^2$ by

$$T((x_1, x_2, x_3, \dots)) = (0, x_1, x_2, \dots).$$

Is T linear? Is it continuous? Is it invertible?

Exercise 0.6. Define a Banach space $D(T) \subset C[0,1]$ such that $T : D(T) \to C[0,1]$ defined by $T(f) = f'$ is well-defined. Is T bounded?

Exercise 0.7. Define the operator $T : L^2[0,1] \to L^2[0,1]$ by

$$T(f)(x) = \int_0^x f(t)\,dt.$$

Is T linear? Is it bounded? Is it invertible?

Exercise 0.8. Let $T : C[0,1] \to C[0,1]$ be defined by

$$T(f)(x) = \int_0^x f(t)\,dt.$$

Show that T is not an open map.

Exercise 0.9. Let $X = Y = L^p[0,1]$, where $1 < p < \infty$, and let $T : X \to Y$ be a bounded surjective linear operator. Prove that the image $T(B_X(0,1))$ contains a ball around 0 in Y.

Exercise 0.10. Let

$$c_{00} = \{x = \{x_n\}_{n\in\mathbb{N}} : \#\{n : x_n \neq 0\} < \infty\}$$

be the space of finitely supported sequences, equipped with the supremum norm

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

For each $n \in \mathbb{N}$, define a linear operator $T_n : c_{00} \to \mathbb{R}$ by

$$T_n(x) = nx_n.$$

(a) Show that the family $\{T_n\}_{n\in\mathbb{N}}$ is pointwise bounded.

(b) Compute the operator norm $\|T_n\|$ for each $n \in \mathbb{N}$. Conclude that the family $\{T_n\}$ is not uniformly bounded.

(c) Explain why this example does not contradict the Banach-Steinhaus theorem.

Exercise 0.11. Let $T : \ell^1 \to \ell^\infty$ be defined by $T(x) = x$. Is that T a bounded operator, does it has a closed graph?

Exercise 0.12. Let $T : D(T) \subset L^2(0,1) \to L^2(0,1)$, with $D(T) = C_c^\infty(0,1)$, and $T(f) = f'$. Determine whether T has a closed graph and whether T is continuous.

Example 0.1. In ℓ^2, the sequence $x_n = (0, 0, ..., 0, 1, 0, ...)$ with 1 in the n-th place converges weakly to 0 but not strongly.

Exercise 0.13. Let $x_n = (1/n, 1/n, 1/n, ..., 1/n, 0, \cdots)$ in ℓ^2, n times $\frac{1}{n}$. Determine whether x_n converges strongly, weakly, or both.

Exercise 0.14. Let $X = \mathbb{R}^N$ for $N \geq 1$. Prove that a sequence $\{x_n\} \subset \mathbb{R}^N$ converges weakly to $x \in \mathbb{R}^N$ if and only if it converges strongly.