Functional Analysis

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Exercise 0.1. Let $e_n = (0, 0, \dots, 1, 0, \dots) \in \ell^\infty = (\ell^1)^*$ be the n-th standard basis vector. Then $e_n \overset{*}{\rightharpoonup} 0$.

Exercise 0.2. Consider the unilateral shift operator $S : \ell^2 \to \ell^2$ defined by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

1. Show that S is a bounded linear operator.
2. Show that S is Fredholm and find its index.

Exercise 0.3. We consider the standard norms in $C[0,1]$ and $C^1[0,1]$:

$$\|g\|_{C[0,1]} = \sup_{x\in[0,1]} |g(x)| \quad \text{for } g \in C[0,1],$$

$$\|f\|_{C^1[0,1]} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| \quad \text{for } f \in C^1[0,1].$$

Define $T : C^1[0,1] \to C[0,1]$ by $Tf = f'$.

1. Show that T is bounded and determine $\ker T$.
2. Describe the range $R(T)$.
3. Show that T is Fredholm and compute its index.

Exercise 0.4. Let $T : \ell^2 \to \ell^2$ be defined by

$$T(x_1, x_2, x_3, \dots) = \left(\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \dots\right).$$

Determine the spectrum $\sigma(T)$ and all eigenvalues of T.

Exercise 0.5. Prove that if T has finite rank, then 0 belongs to the spectrum of T unless T is invertible.

Theorem 0.1 (Arzelà–Ascoli)**.** Let (X, d) be a compact metric space and let $\mathcal{F} \subset C(X)$, where $C(X)$ is the space of continuous real-valued functions on X equipped with the sup norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

Then $\mathcal{F}$ is relatively compact in $C(X)$ (i.e., its closure is compact) if and only if:

1. **Equicontinuity:** For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $x, y \in X$ with $d(x, y) < \delta$,
$$|f(x) - f(y)| < \varepsilon.$$
2. **Pointwise boundedness:** For every $x \in X$, the set
$$\{f(x) : f \in \mathcal{F}\}$$
is bounded in $\mathbb{R}$.

Exercise 0.6. For the operator $T : C[0,1] \to C[0,1]$, defined by

$$(Tf)(x) = \int_0^x f(y)\,dy,$$

prove that T is compact and find all eigenvalues of T.

Exercise 0.7. Let $T : C[0,1] \to C[0,1]$ be defined by

$$(Tf)(x) = \int_0^1 K(x,y)f(y)\,dy,$$

where $K \in C([0,1] \times [0,1])$. Show that T is compact and, if $K(x,y)$ can be written as a finite sum $\sum_{i=1}^m g_i(x)h_i(y)$, then T has finite rank.

Exercise 0.8. Let $T : X \to X$ be a compact operator. Show that $\sigma(T)$ is either finite or countably infinite with 0 as the only possible accumulation point.

Exercise 0.9. Let $T : \ell^2 \to \ell^2$ be given by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots),$$

i.e. the right shift operator. Prove that T is bounded but not compact, and determine its spectrum $\sigma(T)$.

Exercise 0.10. Let $T : \ell^2 \to \ell^2$ be a diagonal operator defined by

$$T(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots),$$

where (λ_n) is a bounded sequence of scalars. Determine when T is compact and describe $\sigma(T)$ in that case.