Functional Analysis

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Exercise 0.1. Let $H = \ell^2$, and let $M \subset H$ be the subspace defined by

$$M = \left\{ x = (x_1, x_2, x_3, \dots) \in \ell^2 : x_n = 0 \text{ for all } n \geq 2 \right\}.$$

That is, $M = \text{span}\{e_1\}$, where $e_1 = (1, 0, 0, \dots)$.
Given the vector $x = (3, 4, 0, 0, \dots) \in \ell^2$, find the orthogonal projection $P_M x$ of x onto M, and compute the distance $\|x - P_M x\|$.

Exercise 0.2. Let $H = L^2([0, 1])$. Define the linear functional

$$\phi(f) = \int_0^1 f(x) \cdot x^2 \, dx.$$

Show that ϕ is a bounded linear functional on H, and find the unique function $g \in H$ such that

$$\phi(f) = \langle f, g \rangle_{L^2} \quad \text{for all } f \in H.$$

Exercise 0.3. Let $H = \ell^2$. Define the linear functional $\phi : H \to \mathbb{C}$ by

$$\phi(x) = 2x_1 - x_3 + ix_4, \quad \text{for } x = (x_1, x_2, x_3, x_4, \dots).$$

Show that ϕ is bounded and find the vector $y \in \ell^2$ such that

$$\phi(x) = \langle x, y \rangle_{\ell^2} \quad \text{for all } x \in \ell^2.$$

Exercise 0.4. Let $T : \ell^2 \to \ell^2$ be the diagonal operator defined by

$$T(x_1, x_2, x_3, \dots) = \left(\frac{1}{1}x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots \right).$$

(a) Show that T is compact and self-adjoint.

(b) Find the spectrum $\sigma(T)$.

(c) Use the spectral theorem to describe an orthonormal basis of eigenvectors for T.

Exercise 0.5. Let $T : L^2([0, 1]) \to L^2([0, 1])$ be the integral operator defined by

$$(Tf)(x) = \int_0^1 \min(x, y) f(y) \, dy.$$

(a) Show that T is compact and self-adjoint.

(b) State what the spectral theorem says about the structure of T.

Exercise 0.6. Let $H = L^2([0, 1])$, and $M = \{f \in H : \int_0^1 f(x)dx = 0\}$. Find the orthogonal complement $M^\perp$.

Exercise 0.7. Let $H = L^2[0, 1]$. Then the sequence

$$\{1\} \cup \left\{ \sqrt{2}\cos(2\pi nt), \sqrt{2}\sin(2\pi nt) \right\}_{n=1}^{\infty}$$

is an orthonormal basis of H.

Exercise 0.8. Let $H = \ell^2$ and define the bilinear form:

$$a(u, v) = \sum_{n=1}^{\infty} \lambda_n u_n v_n,$$

where $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$, and (λ_n) is a bounded sequence. Let $f = (f_1, f_2, \ldots) \in \ell^2$, and define the linear functional:

$$f(v) = \sum_{n=1}^{\infty} f_n v_n.$$

(a) Show that $a(u, v)$ is a bounded bilinear form on ℓ^2.

(b) Show that a is coercive, i.e., $a(u, u) \geq \alpha \|u\|^2$ for some $\alpha > 0$.

(c) Show that $f(v)$ is a bounded linear functional on ℓ^2.

(d) Use the Lax-Milgram theorem to prove that there exists a unique $u \in \ell^2$ such that

$$a(u, v) = f(v) \quad \text{for all } v \in \ell^2.$$

Moreover, find an explicit formula for u.